5. DEGREE PARTITIONS

§5.1. Degree Equations

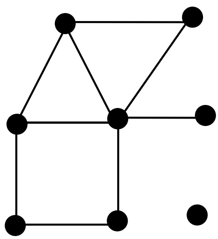
Recall that the **degree** of a vertex v is δ(v), the number of vertices that are adjacent to v or, equivalently, the number of edges that have v as an endpoint. Also, the sum of the degrees is twice the number of edges.

The **degree equation** of a graph is the equation:

$$2E = d_1 + d_2 + \ldots + d_V$$

where V is the number of vertices, E is the number of edges and $d_1 \geq d_2 \geq \ldots \geq d_V$ are the degrees of the vertices.

Example 1: The degree equation of the following graph is 18 = 5 + 3 + 3 + 2 + 2 + 2 + 1 + 0.



We can write it more compactly as:

$$18 = 5 + 3*2 + 2*3 + 1 + 0.$$

Note that with *m*∗*n*, *m* is the degree and *n* is the number of vertices with that degree.

Example 2: The degree equation of K_n is:

$$n^2 - n = (n - 1)*n.$$

The degree equation of $K_{m,n}$ is $2mn = m*n + n*m$.

A fundamental problem is to determine which partitions of the form $2E = \sum_{k=1}^{V} d_k$ where $d_1 \geq d_2 \geq \ldots \geq d_V \geq 0$ are the degree equation for some graph.

Example 3: $10 = 3*3 + 1$ is not the degree equation of any graph. This is because there are 4 vertices and each vertex of degree 3 must be adjacent to all the other vertices, including the vertex of degree 1, which is a contradiction. In general, if $d_V = 1$ then at most one vertex can have degree $V - 1$.

§5.2. The Theorem of Havel and Hakimi

Theorem 1: A partition of the form

$$(1)\ 2E = \sum_{k=1}^{V} d_k$$

where $d_1 \geq d_2 \geq \ldots \geq d_V \geq 0$ is the degree equation for some graph if and only if

$$(2)\ 2(E - d_1) = \sum_{k=2}^{d_1+1} (d_k - 1) + \sum_{k=d_1+2}^{V} d_k$$

is the degree equation for some graph.

Proof: Suppose that (2) is the degree equation of the graph G.

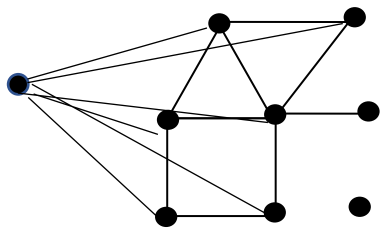
Then add a new vertex and join it to the d_1 vertices whose degrees are largest. (If there are several with the same degree you may need to just arbitrarily choose only some of these to make the total number of vertices adjacent to this new vertex be exactly d_1.

I will complete the proof by proving the converse a little later. But first an example of the above procedure.

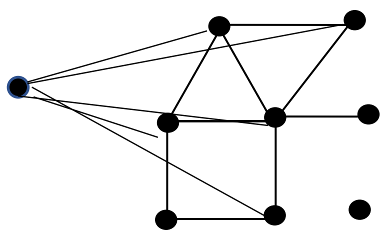
Example 3: The partition:

$$18 = 5 + 3 + 3 + 2 + 2 + 2 + 1 + 0$$

is the degree equation of the graph in example 1. Suppose we wish to have a graph whose largest degree is 6. The degree equation will be obtained by adding 1 to the degrees of the vertices with the 6 largest degrees. The new degree equation will be $24 = 6 + 4 + 4 + 3 + 3 + 3 + 1 + 0$ and this will be the degree equation of the graph:



We could have added a vertex of degree 5 by selecting only 2 of the 3 vertices of degree 2 to connect to the new vertex. This could result in the following:



Proof of Theorem 1 (continued):
Let G be a graph with degree equation (1). If a point of degree d_1 is adjacent to points of degree d_k for k = 1, 2, …, $d_1 + 1$, we merely remove these edges and the resulting graph has degree equation (2). What we need to show is that we are guaranteed to find vertices of degrees d_2 to $d_1 + 1$ that are adjacent to a vertex of degree d_1.

Suppose this is not the case. Let v_i be a vertex of degree d_i in G. Of the vertices of degree d_1 we suppose that v_1 is a vertex of degree d_1 such that the sum of the degrees of vertices adjacent to it is a maximum.

Hence there are points v_i and v_j with $d_i > d_j$ such that v_1 is adjacent to v_j but not v_i.